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Mathematical Problems That Optimize Learning for Academically Advanced Students in Grades K6

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Debate currently exists in the mathematics education community as to the extent that students' needs are being met. Typically, the needs of academically advanced students are not a focus of this discussion. A concern for students of advanced academic capabilities is whether or not sufficient challenge exists to forward genuine understanding of mathematics. Despite Trends in International Mathematics and Science Studies (TIMSS) data (Gonzales et al., 2004; National Center for Educational Statistics, 2009), which indicate that eighth-grade mathematics students do not fare well internationally, fourth-grade mathematics students appear to compete well with peers from other countries on these standardized assessments. However, these data provide a picture of how American students in general are performing, but it does not provide insight with respect to how well or poorly students of advanced academic capabilities are being served. In short, the skills and abilities of advanced students may often be poorly mea-

summary

Several decades ago, V. A. Krutetskii conducted a multiyear study to investigate the various types of thinking that academically advanced, or as he called them, gifted mathematicians used. Following an in-depth look at Krutetskii's nine ways of thinking, a model is proposed that will provide direction for teachers in selecting problems. The model is comprised of four levels of mathematical tasks. Level 1 is mathematical exercises, Level 2 is word or story problems, Level 3 is mathematical problems, and Level 4 is authentic mathematical problem-solving tasks. Subsequently, an elaboration of high- and low-level tasks is applied to the four-level model. Consistent with Krutetskii's theory, the suggestion is then made that approximately ⅓ of the curricula for students of advanced intellect in mathematics should be comprised of Levels 1 and 2 tasks, ⅓ should be comprised of Level 3 tasks, and ⅓ should be comprised of Level 4 tasks. Three implications are offered for teachers and four are offered for researchers. The first implication is that teachers must carefully scrutinize their curriculum to see that it meets the needs of all students, including academically advanced students. The second implication is that conceptual (deep) understanding of algorithms can be attained through the use of mathematical problems and authentically challenging tasks. The third implication is that teachers are not likely to have a database of problems that represents all levels if they use only the provided textbook. Researchers and educators should be reminded that additional time and effort is necessary to empirically research the proposed theory. Moreover, authentically challenging tasks, such as Model-Eliciting Activities, should be used with students, and they could be used for assessment.

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sured in such assessments. Hence, the question remains: Are our most capable students being adequately served in the elementary mathematics classroom?

This paper provides a discussion of how academically advanced mathematics students think. Subsequently, the paper proposes a model or typology for classifying mathematical problems as well as a theory regarding what types of problems should be used with academically advanced students. To conclude the paper, a type of activity known as Model-Eliciting is discussed, as it has been used extensively with academically advanced students.

Literature Review

Vadim A. Krutetskii conducted arguably the most comprehensive study of students who possessed advanced mathematical abilities. During his study, situated in what was then the Soviet Union, Krutetskii (1976) studied and observed students through the use of quantitative and qualitative methods. He compared academically advanced students to average students in an attempt to identify the types of thinking that each exhibited. His intent was to see how academically advanced students' thinking varied from *typical* peers' ways of thinking. Throughout his studies, he was often criticized for using qualitative methods, but this approach ultimately grew to be widely recognized many decades later in the field of educational psychology. Moreover, given the high degree of scrutiny that qualitative research endured, Krutetskii invested great effort in designing the experiments to increase the likelihood that they were highly systematic.

Among the many pieces of data and conclusions to come out of Krutetskii's (1976) studies were nine ways of thinking that academically advanced students possessed that were not possessed by average peers. One caveat is that almost no students possess all nine ways of thinking. In fact, one prospective measure for assessing the level of giftedness of students is to identify the level of expertise in the nine ways of thinking in mathematics. The nine ways of thinking in mathematics are listed below and are detailed in the subsequent sections:

- ability to formalize mathematical material,
- ability to generalize mathematical material,
- ability to operate with numerals and symbols,
- ability to use sequential and logical reasoning (used often in proofs and deductions),
- ability to curtail,
- ability to reverse mental processes,
- ability to think flexibly,
- ability to use mathematical memory, and
- ability to work with spatial concepts.

Ability to Formalize Mathematical Material

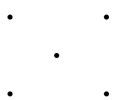
The ability to formalize mathematical material speaks of one's potential to carefully analyze the structure of mathematical problems and to create perceptions based on the structure of the problems. As an example, students may be presented with the task (Krutetskii, 1976), "What, if any, difference exists in the two expressions: $(a - b)^3$ and $a^3 - b^3$?" (p. 236). Individuals with advanced capabilities in formalizing mathematical material will come to the realization that there is a distinct difference in the form of the two expressions and perhaps expand them for proof. When doing problems, interchanging the two expressions can have significant ramifications on the final answer.

Ability to Generalize Mathematical Material

The ability to generalize in mathematics speaks of one's potential to know where and when to apply information to a solution. In any event, one must have the ability to take previously learned material and apply it to novel situations. In a manner of speaking, this ability alludes to transfer and the ability to reason deductively in mathematics. As an example, students in Krutetskii's (1976, p. 241) work were asked to multiply $(C + D + E) \cdot (E + C + D)$ without having an actual algorithm or specific process to successfully solve the problem. Knowing that a process to multiply trinomials was not presented in class, but that the process for multiplying binomials had previously been presented, the observer was seeking to identify whether or not participants could modify, or generalize, the process for binomials to trinomials.

Ability to Operate With Numerals and Symbols

The third ability, to operate with numerals and symbols, aligns closely with the National Council of Teachers of Mathematics (NCTM, 2000) process standard of representation and the content standard of number sense and operations. Individuals who operate with great aplomb and effortlessness while working with numerals and symbols are the focus of this section. The process standard of representation refers to one's ability to move fluently through various representations in mathematics. As an elementary example, numbers can be represented in three principal ways: pictures, words, and symbols.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An individual's ability to work automatically between the three representations in numbers is indicative of one's mathematical abilities. The example of number representation is but one example of representation as they abound in mathematics. As an example, there are four common representations in algebra: text, graph, table, and equation (Van Dyke & Craine, 1997).

Ability to Use Sequential and Logical Reasoning

The ability to use sequential and logical reasoning speaks of one's ability to place items in sequential order and to realize order of events. Regarding logic, there are many applications in mathematics. In elementary grades, a concrete example of logic comes in the realm of set theory in which one may be forced to realize that if C is a subset of B and B is a subset of A, then C must, by definition, be in the larger set of A. For instance,

> Statement 1: Set A contains all great free throw shooters (> 90%) in NCAA basketball.
> Statement 2: All students from school B are great (> 90%) free throw shooters.
> Statement 3: Athlete C is from school B.
> Statement 4: Therefore, student C is a great free throw shooter or in the larger set of A.

This may seem like a trivial example. However, this logic can be applied in myriad ways in mathematics. As an example, this sort of logic may be used with proofs in geometry to make generalizations about the properties of geometric shapes. Hence, the ability to reason sequentially and logically has ostensibly endless applications to mathematics.

Ability to Curtail

The ability to curtail speaks of one's ability to shorten certain processes in an attempt to make the solution more efficient than it otherwise might have been had the entire process been written. Unlike mathematicians who may be intuitive and not capable of explaining why certain steps were chunked together, in curtailment, advanced mathematicians can expand the shortened process when asked to do so. One aspect of curtailment that is quite interesting is that it does not often come immediately upon seeing a problem posed. Curtailment often comes about as a result of seeing a process completed on several occasions and subsequently realizing that curtailment is more efficient than doing the entire process. This is no different than realizing that a shortcut will get one to work quicker than the longer route. When asked to take the longer route, the driver can still complete it although the driver may realize the inefficiency in the expanded route. A mathematical example of curtailment is:

Problem 1: $(x^2 + 8x + 16) = 0$

$(x + 4)\ (x + 4)$

$x = -4$

Problem 2: $(8x^2 + 64x + 128) = 0$

$x = -4$

The student with advanced curtailment capabilities has realized that the second equation is the same as the first equation, but scaled up by a factor of 8. Hence, the student skipped the factoring because the same problem had just been presented, and it would have been highly inefficient to write out all of the steps.

Ability to Reverse Mental Processes

The ability to reverse mental processes refers to working backward to find an answer in problem-solving mode. However, that is not the only use of reversing mental processes. One's ability to reverse a train of thought in mathematics has great implications for one's ability to think flexibly. Krutetskii (1976) described the ability to reverse one's thought process with a diagram of logic. Average students are often capable of thinking in the way that the teacher described (A→B), which could be considered somewhat linear in nature. However, more advanced students hold capabilities that average students do not. With respect to reversibility of thought, advanced students may be inclined to see mathematical procedures in more than one way (e.g., A↔B). A practical example of this could be in the computation of an arithmetic mean. Average students might be able to see that to compute this measure of central tendency one first adds up all numbers in the series and then divides that number by the number of entries in the series. However, more advanced students could complete the aforementioned procedure, but simultaneously be able to identify the number necessary for a specific mean (e.g., find N when 78 + 83 + N + 96 has a mean, $\bar{x}$, of 85).

Ability to Think Flexibly

Closely related to the ability to reverse mental processes is the ability to think flexibly. In fact, the two capabilities are difficult to disentangle in certain instances. One's ability to think flexibly pertains to many things such as the ability to look at an authentic problem-solving task in more than one way. For instance, given a set of data for three airlines, 30 days worth of data, and asked to find the one with the greatest likelihood of being on-time, a purposefully vague term, some students will choose to use the content area of data analysis and probability. Simultaneously, other students will seek to use a number sense and operation perspective. Particularly advanced students may find either approach acceptable and ultimately identify the approach that is most efficient to them. Technically, Krutetskii (1976) defined flexibility in thinking as, "An ability to switch from one mental operation to another; freedom from the binding influence of the commonplace and the hackneyed. This characteristic of thinking is important for the creative work of a mathematician" (p. 88). The final notion in this statement is significant. For mathematicians who are creative, this component is requisite. Moreover, Krutetskii asserts that mathematicians, who truly appreciate the aesthetics of mathematics, often have a high degree of flexibility in thinking. Pragmatically speaking, individuals who are high in flexibility in thinking are not constrained to the single approach that the teacher has provided during class instruction.

Ability to Use Mathematical Memory

Similar to the ability to think flexibly, the ability to use mathematical memory has implications for mathematical problem solving. In specific, the ability to use mathematical memory is used in reference to one's ability to call on long-term and short-term memory (Miller, 1956). One's ability to use mathematical memory deals with the ability to memorize formulae, numbers, and significant material to solve mathematical problems. A simplistic example of one's ability to retain numbers could be a situation in which a number needs to be carried over from one procedure to the next for comparison. The mathematically capable or alert student with a high ability to memorize numbers may not have to revert to a written down number or may not be forced to look up a formula for the subsequent computation. An underlying asset is involved with the ability to use mathematical memory. When low-level procedures need to be completed (e.g., the computation of a standard deviation), a substantial capacity for memory can alleviate cognitive demands on problem solvers thus creating additional cognitive energy for high-level tasks such as self-monitoring. As such, a high capacity for mathematical memory can serve aspiring mathematicians well in situations that demand higher level functioning.

Ability to Work With Spatial Concepts

The ability to work with spatial concepts is similar to the notion of understanding space in mathematics. Some individuals have a greater capacity than others to work with spatial concepts. It is ostensibly the case that these individuals are geometers. However, this is not the only application of spatial reasoning. Individuals with an advanced ability to reason spatially will typically perform well in the NCTM (2000) standard known as measurement, since it has many demands related to assigning a value to two- and three-dimensional figures. Krutetskii (1976) also stated that individuals with advanced spatial reasoning capabilities might have a particular aptitude for engineering.

A Final Caveat

It is not the case that all advanced students will be equally adept in all of the nine areas. One way to consider giftedness in mathematics is to assess students to see where they are on the continuum in all areas. Naturally, a student who has strong aptitudes in several areas would be stronger than a student who has a similar aptitude in only one area. Moreover, the extent to which one has an aptitude in an area speaks of the level of giftedness as well. For instance, if one is assessed and is identified as having the greatest aptitude seen in the area of memory, the pupil may be a prodigy despite having few capabilities in the other areas. To date, no formalized instruments have been validated that use Krutetskii's (1976) theory as a basis. This would be a valuable addition to the field of mathematics. Moreover, such an instrument might provide direction for curriculum developers who identify activities that promote mathematical talent and development.

Activities That Promote Mathematical Talent and Development

The question remains: What type of activities promote deep mathematical understanding for academically advanced mathematics students? An ongoing debate for several decades pertains to the extent to which low-level relative to high-level activities should be used. A follow-up question is, to what extent will low-level activities help academically advanced students develop in light of Krutetskii's nine areas of thinking? Advocates of either position exclusively may be well-served with a dose of moderation. For instance, Lee and Tingstrom (1994) and Woodward (2006), advocates of low-level activities or so-called drill-and-practice in mathematics, suggested that repeated exposure to mathematical procedures produce automaticity. They are correct, but increased levels of automaticity precipitates the question: At the expense of what? In essence, many mathematics educators question whether or not the gains in automaticity (procedural understanding) are worth the decreased attention to conceptual understanding.

Conversely, advocates of high-level activities (Hiebert & Wearne, 1993; Stylianides & Stylianides, 2008) clamor for an exclusive exposure to high-level activities, such as authentic problem-solving tasks. The rationale provided for this approach is that when students are engaged in authentic problem-solving activities, they have access to high- and low-level tasks because doing mathematical problem-solving tasks requires students to do low-level procedural tasks to reach the ultimate goal of successfully solving the mathematical problem-solving task. Their opponents question the extent to which students have access to procedural tasks and often complain that it is not adequate exposure.

NCTM (2000) suggested that a balance needs to be reached between the two extremes. Specifically, it suggested that conceptual understanding and procedural skill knowledge (Davis, 2005; Rittle-Johnson, Siegler, & Alibali, 2001) should be fostered with young mathematicians. Furthermore, they suggest that teaching for understanding (Hiebert et al., 1997; Wood, Merkel, & Uerkwitz, 1996) is a requisite responsibility of mathematics educators. To fully understand what by-products are precipitated by mathematical tasks, a taxonomy for these tasks may be helpful.

Review of Various Types of Mathematical Problems

To gain a deep understanding of the types of problems necessary for academically advanced students in mathematics, it is imperative to comprehend the spectrum of mathematical problems that exist. Before this is done, the caveat is issued that mathematical problem solving could safely be considered a construct. Hence, it is virtually impossible to have the field of mathematics education come to agreement regarding a categorization of *all* types of mathematical tasks, and it may be even more difficult to have the field come to agreement on one definition of mathematical problem solving. Through the use of the Delphi technique, which is a qualitative method designed to seek consensus amongst experts on a topic or construct, it was ascertained that some consensus on the concept of mathematical problem solving exists (Chamberlin, 2008). In a Delphi study, qualitative data from experts is solicited, and it is then turned into quantitative items in which the same group of experts has the opportunity to rate the comments through a Likert scale (Chamberlin, 2008). Subsequently, a model proposed by Chamberlin, Rice, and Chamberlin (2009) is explicated in the forthcoming section in an attempt to provide curricular guidance in

Table 1

Model-Eliciting Activities

Level	Name	Grade Four Example as Task Statements
Level 1	Exercises	478 ÷ 24
Level 2	Word or story problems	J. C. had four apples and Jerome gave him three more. How many does he have now?
Level 3	Mathematical problems	Add the numbers 1 to 73 and tell me what the last two numbers are.
Level 4	Authentic mathematical problem-solving tasks	Using the data presented, identify the best cell-phone plan for your needs and write a rationale for why you have selected it.

decision making for educators of academically advanced mathematicians. This model was derived after an analysis of mathematics curricula in grades K–6 and, like many models, consensus has not been reached by the mathematical education community on it. The model is provided in Table 1.

Exercises

Despite efforts to increase the percentage of mathematical problem-solving tasks in elementary curricula in the United States, it may be common to see an abundance of mathematical exercises done in a typical elementary mathematics classroom. In fact, when parents do not see a great quantity of mathematical exercises assigned to their children each night, some may become concerned because that their children are not *doing* mathematics as per their perception. Their concern rests in the notion that doing mathematics is comprised of doing multiple computations to gain familiarity with algorithms or procedures. Adults not involved in careers in which mathematics is used regularly may be inclined to view mathematics as a set of unrelated rules or procedures as well as skills and knowledge that is optimally learned through rote memorization (Costello, 1991). Examples of tasks that lend themselves to refining skills through mathematical exercises in the elementary grades are addition and its inverse operation, subtraction, and multiplication and its inverse operation, division. As children age, some exercises may involve more than one computation. As an example, to compute an arithmetic mean, one must add up all of the points in a data set and divide by the number of entries. Successful computation of an arithmetic mean therefore requires addition and division (more than one mathematical procedure). After a few attempts, the computation of an arithmetic mean is nothing more than a mathematical exercise. Similarly, regurgitating information on an assessment, such as the identification of shapes and their respective names, is merely an exercise as success in such an endeavor is predicated on rote memorization. It is important to remember that exercises should not be eliminated from mathematical curricula completely as they have a distinct purpose that will be discussed later in the article. In elementary grades, exercises are predominantly used in the field of arithmetic, which the NCTM (2000) refers to as number sense and operations.

Word or Story Problems

A word problem or a story problem can most easily be described as a mathematical exercise surrounded by text. It may be common for educators of the academically advanced to perceive word problems as actual mathematical problem-solving tasks and therefore to presume that cognitive demands are adequate for students. However, assuming students do not have reading difficulties that significantly impede comprehension, word problems do not pose additional cognitive demands on students that exercises do. Students simply seek key words, identify the numbers, and execute the operation. Interestingly enough, in some instances the use of key words may ultimately prove detrimental to success in solving word or story problems (Clement & Bernhard, 2005). As with mathematical exercises, success in solving word or story problems typically involves a great deal of automaticity. That is to say, for success in the mathematical operation, very little cognition actually occurs rather than simply recalling a formula or a fact and executing it with the provided numbers or data.

Mathematical Problems

Unlike exercises and word or story problems, mathematical problems actually involve students engaging in cognition in a novel situation. In regards to the term *novel*, one may consider the notion that students are expected to derive a model or a solution to solve the problem although they have not been provided with one (i.e., a model or solution) explicitly prior to the task. The word *explicitly* is used because students have the knowledge to solve the problem from past mathematical experiences. However, no set formula was pronounced as the solution process to the problem prior to its introduction. Hence, a sense of novelty exists in mathematical problems. The significance of novelty is a component that should not be overlooked in mathematical problems (Chamberlin, 2008). As a counterexample to novelty, when 30 mathematical (identical) exercises are provided as homework with the objective of refining mathematical skills, no novelty exists after two to three problems. Similarly, attaching text to a routine mathematical exercise only makes a word or story problem. Further, mathematical problems are ones that require multiple steps and success in them is not solely predicated on automatically recalling information such as facts, skills, knowledge, procedures, or formulae. Academically advanced students in mathematics may desire novelty simply for challenge, but it has been shown that students in the general population often avoid novelty as it often creates increased anxiety and increased risk for failure (Middleton & Midgley, 2002; Turner et al., 2002).

Authentic Mathematical Problem-Solving Tasks

Authentic mathematical problem-solving tasks are considered, by some, to be the highest level of cognitive challenge for students of all levels and abilities (Lesh & Zawojewski, 2007). This is, of course, with the exception of highly theoretical mathematical problems that may be encountered well beyond instruction in primary and secondary schooling. Some students may not be capable of solving these tasks, and others, not always the academically advanced incidentally, may be successful in solving such tasks. One main difference exists in mathematical problems and authentic mathematical problem-solving tasks. Authentic mathematical problem-solving tasks have a context that has a high degree of realism as opposed to mathematical problems, which may or may not have a context at all. In the event a mathematical problem has a context, it may not be described as well as an authentic one or the context is somewhat contrite. As a result of the realistic nature of authentic mathematical problem-solving tasks, students may be inclined to report a higher level of affect on authentic mathematical problem-solving tasks than they would on mathematical problems. Practically speaking, a high level of affect may translate to greater student interest and persistence in tasks, an identification of great value in the problem, and enhanced self-efficacy in doing the problem.

Balance of HOT and LOT Tasks

Bloom (1956), in his cognitive taxonomy, stated that six levels of cognition existed. Although some debate exists regarding which are higher and which are lower, it is generally agreed that the first two to four levels are lower and the final three to four are higher. In the context of this paper, however, the reader should think about LOT and HOT as not exactly dichotomous terms, but as resting on a continuum.

LOT tasks are those on which the problem solver pulls from previously memorized information. This is often referred to in the field of psychology as the process of automaticity, meaning the problem solver automatically solves the problem without overt cognition. HOT tasks are those on which the problem solver needs to engage in cognition to successfully solve the problem. HOT tasks also have some degree of self-regulation in monitoring the level of success in problem solving.

On the surface, some mathematics educators may suggest that academically advanced students should engage exclusively in authentic mathematical problem-solving tasks. This decision could prove detrimental to the development of academically advanced students in mathematics as they need to have a mix of low- and high-level skills. With a typical curriculum in the U.S., proficiency in low-level tasks is not a concern as it may be the predominance of activities (Chamberlin et al., 2009). Examples of low-level activities such as worksheets, flash cards, computer software designed to enhance drill-and-practice ability, as well as timed tests may help students polish what are known as "math facts" and these are quite prevalent. High-level tasks, such as authentic mathematical problem-solving activities, often prove the more difficult undertaking in elementary mathematics classrooms due to time required to implement them (Hiebert & Wearne, 1993) and their accessibility (Stylianides & Stylianides, 2008). It is important to note that basic exercises need not be the same for academically advanced students as they are for peers in the general population; they can increase in difficulty a great deal. Moreover, it should not be the case that more of the same activity is assigned to advanced students. Educators need to invest significant time in differentiation of curriculum with academically advanced students so that they can be challenged with exercises. As an example, if first-grade students are busy memorizing their addition facts, academically advanced students may be working on their multiplication facts or even learning about roots or radicals.

Perhaps the most compelling reason why a balance of activities must be sought is so problem solvers will have sufficient cognitive energy to successfully solve complex mathematical tasks. Cognitive energy has also been referred to as mental energy (Lykken, 2005). Cognitive energy works in much the same way that metabolic energy does. Each person possesses a finite amount of each type of energy during a finite period of time. Metabolic energy is measured by the number of calories that one can expend. Neuropsychologists who are interested in cognitive energy, however, have not devised a method to quantify expenditures in cognitive energy (Lykken, 2005). Nevertheless, the principle can be applied to each type of energy. The more energy that one expends in an activity, the less energy is available to expend in another activity. As an example, a fourth-grade academically advanced mathematician is doing a problem such as Departing On Time (see http://crlt.indiana.edu/research/csk.html). During this task, the student needs to donate significant cognitive energy to computing measures of central tendency such as mean, median, and mode, due to a lack of automaticity or understanding of algorithms. As a result of the significant time donated to compute the measures of central tendency, the student is less able to donate significant energy to metacognitive functions that are higher level and may help the student successfully execute tasks such as identifying the most appropriate answer, making sure the solution is the most efficient one, and/or staying on task. When a hiatus in concentration occurs, it may be a result of a lack of cognitive energy. Hence, enabling students to reach automaticity with ostensibly elementary math facts or operations, such as addition, subtraction, multiplication, and division, may enable students the opportunity to invest increased amount of cognitive energy to more demanding tasks. Curricula therefore should have a balance of tasks with low- and high-level demands. The word balance is not intended to suggest a 50 (LOT) and 50 (HOT) mixture.

Optimal Percentage of HOT and LOT Tasks

Given this information, a more precise question and a logical concern is the amount of time that should be invested in respective LOT and HOT tasks. Making a blanket statement for all classroom teachers and academically advanced students would be arrogant and presumptuous. One answer will not suffice for the needs of all learning situations. However, some generalizations could prove beneficial for educators of academically advanced students in mathematics. The first two levels should receive no more than 33% of the curriculum emphasis in the elementary grades. Because the two tasks are extremely similar in nature, it is not important to seek a quantitative value for each type of task. Exercises or word or story problems should not be neglected exclusively, but teachers may find exercises to be a more efficient use of time given the low-level nature of word or story problems. Fundamentally, word or story problems are simply exercises disguised with words. As such, being able to decode math problems from text is an important responsibility of burgeoning mathematicians, but text may slow down the progress of intellectually advanced students in doing a series of problems when the goal is automaticity. The idea of using low-order thinking tasks as no more than 33% of the curriculum is based on Krutetskii's (1976) notion that students of promise learn and memorize mathematical facts at a faster rate than their peers. Ergo, given the fact that promising students learn or memorize at a quicker pace than peers, it may be logical to minimize the use of such activities so that additional efforts and time may be donated to more academically challenging and engaging activities. In the end, increasing the emphasis on HOT activities provides academically advanced students with a realistic portrait of what actual mathematicians do in their vocation.

With approximately 33% of the curriculum invested in low-level activities, approximately 67% of the remaining curriculum can be invested in mathematical problems and authentic mathematical problem-solving tasks, which could be divided evenly between the two emphases. A simple way to think about the allocation of LOT and HOT tasks in mathematics is that one third of math work could be invested in exercises and word or story problems, one third could be invested in mathematical problems, and the final one third could be invested in authentic mathematical problem-solving tasks. The principal reasons for such emphases are that the final two types of tasks engage students in HOT and they mimic what real-life mathematicians do. Richard Lesh and colleagues (2000) referred to activities such as engaging students in precollege-level mathematics, and they suggest that Model-Eliciting Activities (MEAs) are examples of such activities. MEAs are activities that ask students to create models to explain mathematical phenomena. They were initially created for use with general population students and have since been adopted and adapted for gifted students in mathematics. Several publications exist in which the use of MEAs is discussed with academically advanced students (Chamberlin, 2002; Chamberlin & Moon, 2005, 2008; Sriraman, 2005). Because Model-Eliciting Activities have shown great promise in educating academically advanced students, a close analysis of their structure may suggest that they match closely with the nine ways of thinking that Krutetskii (1976) identified.

Implications

Teachers Must Take a Close Look at Curricula in Use

Teachers may choose to alter the proposed model to fit their needs. Nonetheless, teachers are likely to find the model quite applicable to their classroom as it has been designed from literature, textbook analysis, and classroom experience.

As such, the first implication is that teachers and parents need to closely scrutinize the adopted curriculum for students or children of advanced intellect in mathematics. Acceptance from the textbook adoption committee is not insurance that needs of academically advanced students in mathematics will be met. It is quite probable that an overreliance on low-level tasks, such as exercises and word or story problems, is inherent in the curriculum. This may be the result of textbook companies writing texts to prepare students for state standardized tests. Such tests are helpful in assessing the majority of students, but their use with academically advanced students has not been empirically tested.

Further, having a strong knowledge of mathematical facts is rarely criticized unless it comes at the expense of having HOT skills. In other words, some practice with mathematical problem-solving tasks is requisite to be able to do them successfully in the future. Ergo, students must be provided with serious challenges that enable them to utilize HOT skills. Without significant access to HOT tasks, the potential by-product is that academically advanced mathematics students may be inclined to become bored; have negative affect, such as attitude, interest, and value about mathematics; and become disengaged and less persistent with excessive exercises and word problems (Chamberlin, 2002). When negative affective ratings are maintained for extended periods of time, such as several months, temporary emotions run the risk of becoming permanent. One less-than-capable mathematics instructor for a year may permanently damage an aspiring student's mathematics affect for life.

Conceptual Understanding of Algorithms and Authentically Challenging Tasks Are Needed

One of the most influential and pragmatic books for mathematics educators was written near the end of the last millennium (Hiebert et al., 1997). In this publication, Hiebert and colleagues (1997) stressed the notion of making sense of mathematics. This would appear to be an emphasis of all mathematics classrooms, although reality may prove otherwise. How does one not concentrate on making sense of mathematics? This can be done by simply concentrating on the algorithms without any concern for why they work as they do. This may be a familiar tactic to teachers with poor content knowledge. When "why" questions are used, the true meaning of mathematics may surface. Consequently, if academically advanced mathematicians are to be challenged in mathematics, it is incumbent upon mathematics teachers in elementary grades to help students consider why certain procedures have been accepted by the mathematics community as the most efficient method available.

Mathematics Textbook Tasks May Not Suffice

Perhaps the teaching method that requires the least effort is to open the textbook and use the problems for the lesson. However, many elementary textbooks may not have a satisfactory number of authentically challenging tasks for students of advanced academic capabilities, although some exceptions appear to exist (Chamberlin et al., 2009; Stlyianides & Stlyianides, 2008). These textbooks may not even have authentically challenging tasks for students in the general population. Elementary math textbooks may be a compilation of exercises and word problems, which are very helpful if the objective is to hone low-level skills. Despite what is advertised and printed in books, legitimate mathematical problem-solving tasks may be sparse in elementary math textbooks. For this reason, teachers of academically advanced students may need to look elsewhere to find resources to genuinely challenge students.

Areas for Future Research

All theories must be tested at one time or another. To that end, the proposed levels are simply proposed levels until and unless researchers test the theory. Actual data must be solicited to see if the levels are theoretically sound. This could be done in several manners, and perhaps the most expeditious of these approaches would be to conduct a Delphi study to seek input from experts in mathematical curricula.

Given the fact that no formalized instrument for use with academically advanced students has been created with Krutetskii's (1976) theory as a basis, this too would be an area for future research. Such an instrument could aid the field of mathematics education greatly in understanding academically advanced students. In addition, it would provide a venue for systematically identifying academically advanced mathematicians with a formalized instrument.

In addition, the link between Model-Eliciting Activities and Krutetskii's (1976) nine ways of thinking should be explored. This relationship was only cursorily referenced in this publication, and it deserves further attention from an empirical perspective. MEAs provide great potential for challenging academically advanced students, and if it could be illustrated that they provide promise through research, they would warrant significant consideration among educators of academically advanced students.

As a final area for future research, it would behoove educational psychologists to use MEAs as an instrument to identify academically advanced students. In the second suggestion, a cry for formalized instruments was echoed. In the third suggestion, it was suggested that MEAs be used with academically advanced students. To incorporate the two areas, a fourth suggestion was created, which is to use MEAs to identify academically advanced mathematicians. How this could be done remains somewhat unclear, but potential does exist if this notion could be conceptualized.

Conclusion

The status quo can be maintained and academically advanced mathematics students can continue to be educated in the manner in which they have for several decades. However, this mode of instruction is not likely to adequately challenge the most promising students. Moreover, a case has been outlined for taking a close look at teaching and learning practices in elementary classrooms. Given the fact that curricula often greatly inform teaching and learning practices, perhaps the best place to start is by carefully analyzing mathematics curricula. The curricula in many classrooms may have an overreliance on routine procedures and low-level skills. This overreliance may come through the use of textbook-based problems, which are likely mathematical exercises or word or story problems. In short, these types of tasks often have a focus on low-level thinking skills, which are not often a need for academically advanced students in mathematics.

To fully optimize learning situations for such students, high-level mathematical problems must be employed. In employing HOT mathematical tasks, cognitive demands of academically advanced students have a greater likelihood of being met relative to using LOT mathematical tasks. MEAs appear to be one such type of supplementary curriculum that has shown promise in adequately challenging academically advanced students, and they appear to have strong relationships to the problems used in high-performing countries such as Singapore, China, and Japan (Leung, 2005).

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